Nonlocal Cosmological Models with Quadratic Potentials. Localization and Exact Solutions

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based on

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To specify different types of cosmic fluids one uses a phenomenological relation between the pressure p and the energy density ϱ

$$p = w\varrho, \qquad p = E_k - V, \qquad \varrho = E_k + V$$

where w is the state parameter.

$w > 0$ — **Atoms (4%)**
$w = 0$ — **the Cold Dark Matter (23%)**
$w < 0$ — **the Dark Energy (73%)**

$$w(t) = \frac{p}{\varrho} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = -1 + \frac{2E_k}{\varrho}. \tag{1}$$

Contemporary experiments give strong support that

$$w_{DE} \approx -1. \tag{2}$$

We consider the case $w_{DE} < -1$. Null energy condition (NEC) is violated and there are problems of instability. A possible way to evade the instability problem for models with $w_{DE} < -1$ is to yield a phantom model as an effective one, arising from a more fundamental theory.

In particular, if we consider a model with higher derivatives such as

$$\phi e^{-\Box_g}\phi, \tag{3}$$

then in the simplest approximation:

$$\phi e^{-\Box_g}\phi \simeq \phi^2 - \phi\Box_g\phi, \tag{4}$$

such a model gives a kinetic term with a ghost sign. Such a possibility does appear in the string field theory framework:

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1 Nonlocal model

We consider a model of gravity coupling with a nonlocal scalar field, which induced by strings field theory

$$S=\int d^4x\sqrt{-g}\left(\frac{m_p^2}{2}R+\frac{\xi^2}{2}\phi\,\Box_g\phi+\frac{1}{2}\Big(\phi^2-c\,\Phi^2\Big)-\Lambda\right), \tag{5}$$

$$\Phi=e^{\Box_g}\phi, \tag{6}$$

where g is the metric,

$$\Box_g=\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu.$$

$m_p^2=g_4M_p^2/M_s^2$, where

M_p is a mass Planck,
M_s is a characteristic string scale,
g_4 is a dimensionless effective coupling constant.

Λ is an effective cosmological constant.
ξ and c are positive constants.

2 Roots of the Characteristic Equation

Let us consider equation of motion for ϕ:

$$\mathcal{F}(\Box_g) = (\xi^2 \Box_g + 1) e^{-2\Box_g} \phi = c\, \phi. \tag{7}$$

We assume that the metric $g_{\mu\nu}$ is given and consider eq. (7) as an equation in ϕ.

The eigenfunctions of the Beltrami-Laplace operator

$$\Box_g \phi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \phi = M\phi, \tag{8}$$

also represent the solutions of equation of motion (7) with M, which is defined as a solution of the characteristic equation

$$\xi^2 M + 1 - c\, e^{2M} = 0. \tag{9}$$

The characteristic equation does not depend on metric!

This result does not depend on form of $\mathcal{F}$.

3 Generalization

Let us consider the more general case:

The function $\mathcal{F}$ is assumed to be an analytic function, which can be represented by the convergent series expansion:

$$\mathcal{F} = \sum_{n=0}^{\infty} f_n \Box_g^n \text{ and } f_n \in \mathbb{R}. \tag{10}$$

This case is described by the following action:

$$S = \int d^4x \sqrt{-g} \left(\frac{m_p^2}{2} R + \frac{1}{2} \phi \mathcal{F}(\Box_g) \phi + \Lambda \right). \tag{11}$$

Equations of motion are

$$\mathcal{F}(\Box_g)\phi = 0 \tag{12}$$

$$G_{\mu\nu} = \frac{1}{m_p^2}(T_{\mu\nu} + \Lambda), \tag{13}$$

where

$$T_{\mu\nu} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{l=n-1} \Big(\partial_\mu \Box_g^l \phi \partial_\nu \Box_g^{n-1-l} \phi + \partial_\nu \Box_g^l \phi \partial_\mu \Box_g^{n-1-l} \phi - \\ -g_{\mu\nu} \left(g^{\rho\sigma} \partial_\rho \Box_g^l \phi \partial_\sigma \Box_g^{n-1-l} \phi + \Box_g^l \phi \Box_g^{n-l} \phi \right) \Big), \tag{14}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the energy-momentum (stress) tensor.

One can take the following function as a solution

$$\phi = \sum_i^N \phi_i \quad \text{where} \quad \Box\phi_i = M_i\phi_i \quad \text{and} \quad \mathcal{F}(M_i) = 0. \tag{15}$$

Without loss of generality we assume that for any i_1 and $i_2 \neq i_1$ condition $M_{i_1} \neq M_{i_2}$ is satisfied.

In an arbitrary metric the energy-momentum tensor evaluated on such a solution is

$$T_{\mu\nu} = \sum_i \mathcal{F}'_{,M}(M_i)\left(\partial_\mu\phi_i\partial_\nu\phi_i - \frac{1}{2}g_{\mu\nu}\left(g^{\rho\sigma}\partial_\rho\phi_i\partial_\sigma\phi_i + M_i\phi_i^2\right)\right) \tag{16}$$

Note that all terms, which proportional to $\phi_i\phi_j$ or $\partial_\mu\phi_i\partial_\nu\phi_j$ are equal to zero at $i \neq j$. To obtain this we use the condition that $\mathcal{F}(M_i) = 0$.

The spatially flat Friedmann–Robertson–Walker (FRW) metric is

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right) \tag{17}$$

where $a(t)$ is the scale factor. The fields ϕ are taken to be space-homogeneous as well. The energy-momentum tensor in this metric can be written in the form of a perfect fluid

$$T^{\mu}_{\nu} = \mathrm{diag}(-\varrho, p, p, p),$$

where

$$\begin{aligned} \varrho &= \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{l=n-1}\left(\partial_t \Box_g^l \tau \partial_t \Box_g^{n-1-l}\tau + \Box_g^l \tau \Box_g^{n-l}\tau\right), \\ p &= \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{l=n-1}\left(\partial_t \Box_g^l \tau \partial_t \Box_g^{n-1-l}\tau - \Box_g^l \tau \Box_g^{n-l}\tau\right). \end{aligned} \tag{18}$$

Friedmann equations are

$$3m_p^2H^2 = \varrho + \Lambda, \quad 2m_p^2\dot{H} = -(\varrho + p). \tag{19}$$

The consequence of (19) is the conservation equation:

$$\dot{\varrho} + 3H(\varrho + p) = 0. \tag{20}$$

Note that system (19) is a non-local and non-linear system of equation. At the same time using formulae for the energy density and pressure it is possible to generate local systems out of (19), corresponding to particular solutions of the initial non-local system.

Let us calculate the energy density and the pressure for

$$\phi = \sum_{k=1}^{N} \phi_k, \tag{21}$$

where $\Box\phi_k = M_k\phi_k$ and $\mathcal{F}(M_k) = 0$ for all k.

We obtain:

$$\begin{aligned} \varrho(\phi) &= \frac{1}{2}\sum_{k=1}^{N} \mathcal{F}'(M_k)\left(\dot{\phi}_k^2 + M_k\phi_k^2\right) \\ p(\phi) &= \frac{1}{2}\sum_{k=1}^{N} \mathcal{F}'(M_k)\left(\dot{\phi}_k^2 - M_k\phi_k^2\right) \end{aligned} \tag{22}$$

System (19) is as follows:

$$\begin{cases} 3m_p^2 H^2 = \sum_k \mathcal{F}'(M_k)\left(\dot{\phi}_k^2 + M_k\phi_k^2\right) + \Lambda, \\ 2m_p^2 \dot{H} = \sum_k \mathcal{F}'(M_k)\dot{\phi}_k^2. \end{cases} \tag{23}$$

We also have

$$\Box_g \phi_k = (-\partial^2 - 3H\partial)\phi_k = M_k\phi_k, \qquad k = 1, 2, \ldots, N. \tag{24}$$

We obtain the Einstein equation for the following action:

$$S_{local} = \int d^4x\sqrt{-g}\left(\frac{m_p^2}{2}R + \right.$$

$$\left. + \sum_k \mathcal{F}'(M_k)\left(-g^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k - M_k\phi_k^2\right) - \Lambda\right).$$

4 Real roots of characteristic equation

Let as consider the SFT inspired action with

$$\mathcal{F}(\Box_g) = (\xi^2\Box_g + 1)e^{-2\Box_g}\phi = c\,\phi, \tag{25}$$

$$\Box_g\phi = M\phi \equiv \alpha^2\phi. \tag{26}$$

The characteristic equation

$$-\xi^2\alpha^2 + 1 - c\,e^{-2\alpha^2} = 0. \tag{27}$$

has the following solutions

$$\alpha_n = \pm\frac{1}{2\xi}\sqrt{4 + 2\xi^2 W_n\left(-\frac{2ce^{-2/\xi^2}}{\xi^2}\right)}, \quad n = 0, \pm1, \pm2, \ldots \tag{28}$$

where W_n is the n-s branch of the Lambert function satisfying a relation $W(z)e^{W(z)} = z$.

To determine values of the parameters at which eq. (27) has real roots $m = \alpha$ we rewrite this equation in the following form:

$$\xi^2 \equiv \xi^2(\alpha, c) = \frac{e^{2\alpha^2} - c}{\alpha^2 e^{2\alpha^2}} = \frac{e^{2m^2} - c}{m^2 e^{2m^2}}. \qquad (29)$$

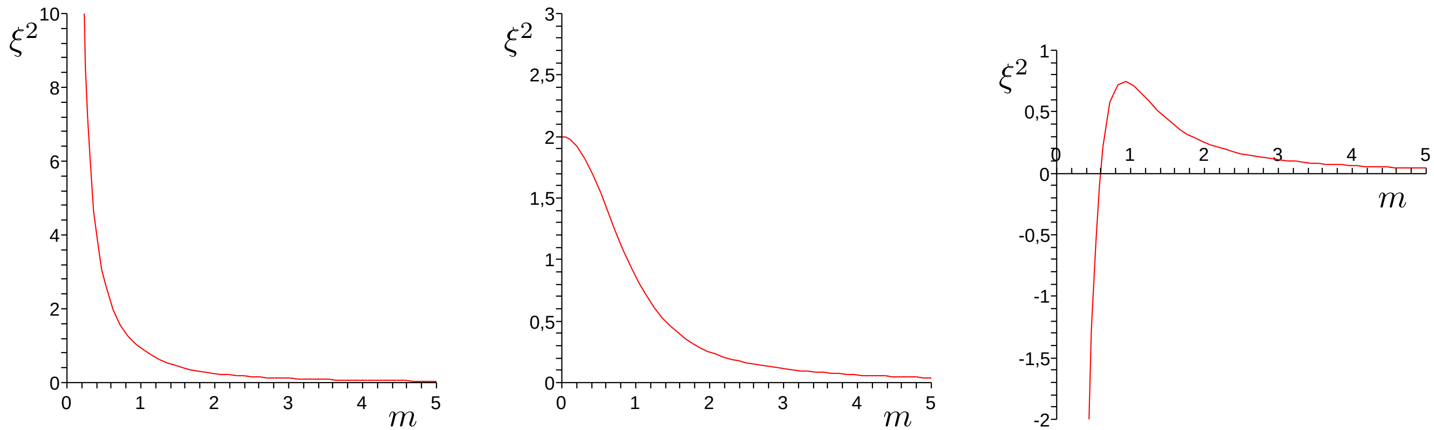


Figure 1: The dependence of ξ^2 on $m = \alpha$ at $c = 1/2$ (left), $c = 1$ (center) and $c = 2$ (right).

The function $\xi^2(\alpha)$ has a maximum provided c is such that $W_{-1}\left(-\frac{e^{-1}}{c}\right) < -1$, in the other words $0 < c < 1$.

- If $c < 1$, then (27) has two simple real roots for any ξ.
- If $c = 1$, then (27) has a zero root. Nonzero real roots exist at $\xi^2 < 2$.
- If $c > 1$, then (9) has no real roots for $\xi^2 > \xi^2_{max}$, 2 double roots for $\xi^2 = \xi^2_{max}$, and 4 real simple roots for $\xi^2 < \xi^2_{max}$.

$$\varrho(\phi_1) = \frac{\eta_{\alpha_1}}{2}\left(\left(\dot{\phi}_1\right)^2 - \alpha_1^2\phi_1^2\right), \tag{30}$$

$$p(\phi_1) = \frac{\eta_{\alpha_1}}{2}\left(\left(\dot{\phi}_1\right)^2 + \alpha_1^2\phi_1^2\right), \tag{31}$$

where for arbitrary α $$\eta_\alpha \equiv \xi^2 + 2\xi^2\alpha^2 - 2. \tag{32}$$

If and only if $c > 1$, then there exists the interval of $0 < m^2 < m^2_{max}$, on which $\eta_m < 0$. Some part of this interval is not physical, because $g(m^2, c) < 0$ on this part. The straightforward calculations show that at the point

$$m^2_{max} = -\frac{1}{2} - \frac{1}{2} W_{-1}\left(-\frac{e^{-1}}{c}\right), \qquad (33)$$

we obtain $\eta_m(m_{max}) = 0$. So, for $c > 1$ and $\xi^2 < \xi^2_{max}$ we have two positive roots of (27): m_1 and $m_2 > m_1$, with $\eta_{m_1} < 0$ and $\eta_{m_2} > 0$.

In the case of two real roots $\alpha_1 > 0$ and $\alpha_2 > \alpha_1$:

$$\begin{cases} 3H^2 = \frac{1}{2m_p^2}\left(\eta_{\alpha_1}\left(\dot\phi_1^2 - \alpha_1^2\phi_1^2\right) + \eta_{\alpha_2}\left(\dot\phi_2^2 - \alpha_2^2\phi_2^2\right) + \Lambda'\right), \\ \dot H = -\frac{1}{2m_p^2}\left(\eta_{\alpha_1}\dot\phi_1^2 + \eta_{\alpha_2}\dot\phi_2^2\right), \end{cases}$$

we have obtained that $\eta_{\alpha_1} < 0$ and $\eta_{\alpha_2} > 0$. Therefore the corresponding two-field model is a quintom one.

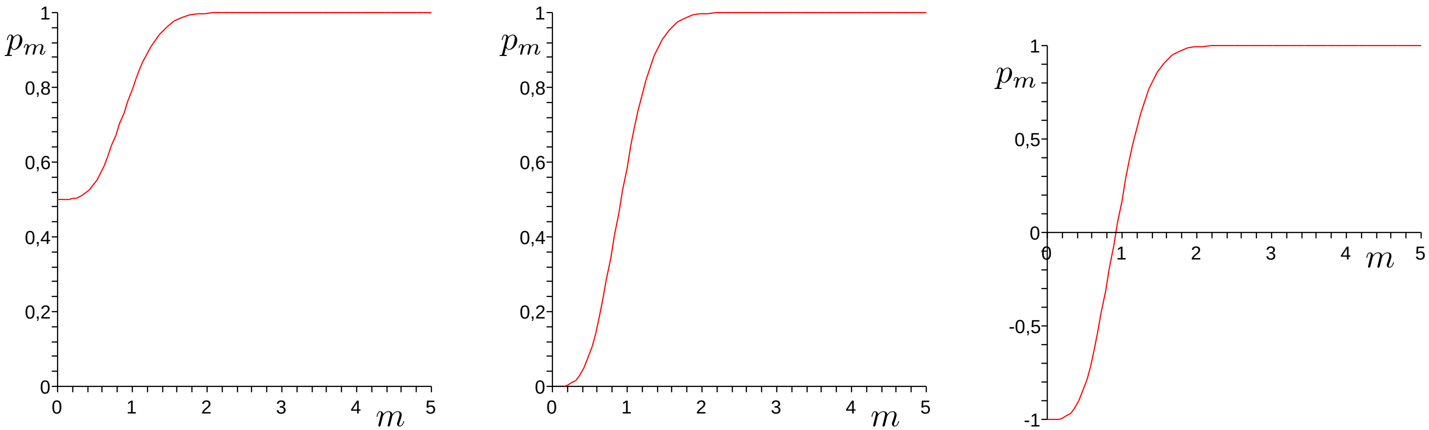


Figure 2: The function $p_m = m^2 \eta_m$ at $c = 1/2$ (right), $c = 1$ (center) and $c = 2$ (left).

5 Exact Solution in the case $N = 1$

At present time one of the possible scenarios of the Universe evolution considers the Universe to be a D3-brane (3 spatial and one time variable) embedded in higher-dimensional space-time. This D-brane is unstable and does evolve to the stable state. A phantom scalar field is an open string theory tachyon. According to the Sen's conjecture this tachyon describes brane decay, at which a slow transition in a stable vacuum takes place.

We assume that the phantom field $\phi(t)$ smoothly rolls from the unstable perturbative vacuum ($\phi = 0$) to a nonperturbative one $\phi = A_0 \neq 0$ and stops there. In other words we seek a kink-type solution.

At $c = 1$ one of solutions of eq. (9) is $\alpha = 0$ and we obtain the following system

$$\begin{cases} 3H^2 = \dfrac{\eta\alpha}{2m_p^2}\dot{\phi}^2 + \dfrac{\Lambda}{m_p^2}, \\ \dot{H} = -\dfrac{\eta\alpha}{2m_p^2}\dot{\phi}^2. \end{cases} \tag{34}$$

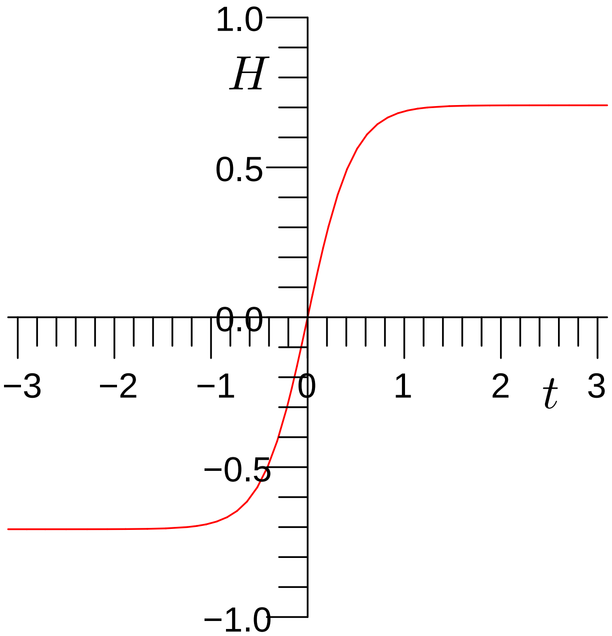
If $\Lambda > 0$, then there exist the following real solution:

$$H_1(t) = \sqrt{\frac{\Lambda}{3m_p^2}}\tanh\left(\sqrt{\frac{3\Lambda}{m_p^2}}(t-t_0)\right), \tag{35}$$

$$\phi(t) = \pm\sqrt{\frac{2m_p^2}{3(2-\xi^2)}}\arctan\left(\sinh\left(\sqrt{\frac{3\Lambda}{m_p^2}}(t-t_0)\right)\right) + C_1,$$

where t_0 and C_1 are arbitrary constants.

The Hubble parameter $H(t)$ is a monotonically increasing function, so $w < -1$.



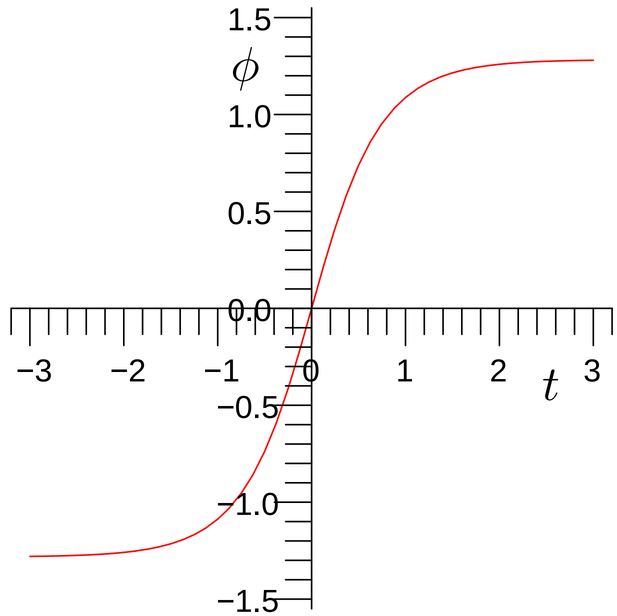


Figure 3: The functions $H_1(t)$ (right) and $\phi_1(t)$ (left) at $\Lambda = 3/2$, $m_p^2 = 1$, $\xi^2 = 1$, $t_0 = 0$ and $C_2 = 0$.

$$E = \varrho + \Lambda = \Lambda \left(1 - \frac{1}{\cosh\left(\sqrt{\frac{3\Lambda}{m_p^2}}(t - t_0)\right)^2} \right) \tag{36}$$

$$P = p - \Lambda = \Lambda \left(-1 - \frac{1}{\cosh\left(\sqrt{\frac{3\Lambda}{m_p^2}}(t - t_0)\right)^2} \right) \tag{37}$$

$$w = -1 - \frac{2}{\cosh\left(\sqrt{\frac{3\Lambda}{m_p^2}}(t - t_0)\right)^2 - 1} \tag{38}$$

Note that we have found two-parameter set of exact solutions at any $\Lambda > 0$. In other words, we have found the general solution.

Another solution

$$\tilde{H}_1(t) = \sqrt{\frac{\Lambda}{3m_p^2}} \coth\left(\sqrt{\frac{3\Lambda}{m_p^2}}(t - t_0)\right), \tag{39}$$

It is interesting that type of solution essentially depends on sign of Λ.

At $\Lambda = 0$ and $\eta_\alpha = \xi^2 - 2 > 0$

$$\phi(t) = \pm\sqrt{\frac{2m_p^2}{3\eta_\alpha}}\ln(t - t_0) + C_1, \qquad H(t) = \frac{1}{3(t - t_0)}, \quad (40)$$

where t_0 and C_1 are arbitrary constants.

In the case $\Lambda < 0$ we obtain the following general solution:

$$H(t) = -\sqrt{\frac{-\Lambda}{3m_p^2}}\tan\left(\sqrt{-\frac{3\Lambda}{m_p^2}}(t - t_0)\right), \qquad (41)$$

$$\phi(t) = \pm\sqrt{\frac{8m_p^2}{3(\xi^2 - 2)}}\text{arctanh}\left(\frac{\cos\left(\sqrt{\frac{-3\Lambda}{m_p^2}}(t - t_0)\right) - 1}{\sin\left(\sqrt{\frac{-3\Lambda}{m_p^2}}(t - t_0)\right)}\right) + C_1.$$

6 Solutions in the Bianchi I metric

In Bianchi I metric with the interval

$$ds^2 = -dt^2 + a_1^2(t)dx_1^2 + a_2^2(t)dx_2^2 + a_3^2(t)dx_3^2, \qquad (42)$$

the Einstein equations has the following form:

$$H_1H_2 + H_1H_3 + H_2H_3 = \frac{1}{m_p^2}\left(\frac{\eta\alpha}{2}\dot{\phi}^2 + \Lambda\right), \qquad (43)$$

$$\dot{H}_2 + H_2^2 + \dot{H}_3 + H_3^2 + H_2H_3 = -\frac{1}{m_p^2}\left(\frac{\eta\alpha}{2}\dot{\phi}^2 - \Lambda\right), \qquad (44)$$

$$\dot{H}_1 + H_1^2 + \dot{H}_2 + H_2^2 + H_1H_2 = -\frac{1}{m_p^2}\left(\frac{\eta\alpha}{2}\dot{\phi}^2 - \Lambda\right), \qquad (45)$$

$$\dot{H}_1 + H_1^2 + \dot{H}_3 + H_3^2 + H_1H_3 = -\frac{1}{m_p^2}\left(\frac{\eta\alpha}{2}\dot{\phi}^2 - \Lambda\right), \qquad (46)$$

where $H_k \equiv \dot{a}_k/a_k$, $k=1,2,3$.

Our goal is to present exact solutions to system (43)–(46). Of course, there exist isotropic solutions, which coincide with exact solutions in the FRW metric. For those solutions $H_1(t) = H_2(t) = H_3(t)$. At the same time exact anisotropic solutions do exist.

Let us consider the case of $\Lambda = m_p^2 > 0$.

There exist not only isotropic solution

$$H_1(t) = H_2(t) = H_3(t) = \frac{1}{\sqrt{3}} \tanh\left(\sqrt{3}(\tilde{t})\right), \qquad (47)$$

but also an anisotropic one

$$
\begin{aligned}
H_1(t) &= \frac{1}{\sqrt{3}} \tanh\left(\frac{\sqrt{3}}{2}\tilde{t}\right), \\
H_2(t) &= \frac{1}{\sqrt{3}} \coth\left(\frac{\sqrt{3}}{2}\tilde{t}\right), \\
H_3(t) &= \frac{1}{2\sqrt{3}} \left(\tanh\left(\frac{\sqrt{3}}{2}\tilde{t}\right) + \coth\left(\frac{\sqrt{3}}{2}\tilde{t}\right)\right).
\end{aligned} \tag{48}
$$

The corresponding scalar field is real at $B > 0$

$$
\tilde{\phi}(t) = \frac{2m_p}{3\sqrt{\eta\alpha}} \left(\ln(e^{\sqrt{3}\tilde{t}} + 1) - \ln(e^{\frac{\sqrt{3}}{2}\tilde{t}} - 1) - \ln(e^{\frac{\sqrt{3}}{2}\tilde{t}} + 1)\right),
$$

where $\tilde{t} = t - t_0$.

Conclusions

We have studied the SFT inspired linear nonlocal model, which is characterized by two parameters: ξ^2 and c and obtained:

1. Roots of the characteristic equation do not depend on the form of the metric.

2. In an arbitrary metric the energy-momentum tensor for an arbitrary N-mode solution is a sum of the energy-momentum tensors for the corresponding one-mode solutions.

3. In the FRW spatially flat metric the pressure for a one-mode solution corresponding to a real root (at $c > 1$) can be positive or negative.

4. Our linear model with one nonlocal scalar field generates an infinite number of local models. Hence, special solutions for nonlocal model in the FRW metric can be obtained.

5. We have constructed (arXiv:0711.1364) an exact kink-like solution, which correspond to monotonically increasing Universe with phantom dark energy.

6. We have constructed an exact solution in the Bianchi I metric.

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