Noncommutative deformations of quantum field theories, locality and causality

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Motivation and Program

Restrictions imposed by the uncertainty principle and gravity on measurements

Low-energy limit of string theory

$$\Downarrow$$

$$[x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$$

$$\Uparrow$$

Star products: Moyal-Weyl-Grönewold, Wick-Voros

$\uparrow$

(Galluccio, Lizzi, Vitale, (2008))

$\Downarrow$ **(Balachandran, Martone, (2009))**

Twisted Poincaré covariance and twisted tensor product $\otimes_\star$

$$\Downarrow$$

Under what conditions do the star and twisted tensor products converge?

How should the causality principle be implemented in NC QFT?

Plan

- Introduction: Star products and twisted tensor products
- Noncommutative deformations of quantum field theories
- Two concepts of wedge-locality
- Convergence of star products and adequate function spaces
- θ-Locality instead of microcausality
- Conclusions

Moyal $\star$-product

$$(f \star_M g)(x) = f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial_\mu}\, \theta^{\mu\nu}\, \overrightarrow{\partial_\nu}\right) g(x)$$

$$= f(x)g(x) + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\mu_1\nu_1} \ldots \theta^{\mu_n\nu_n} \partial_{\mu_1} \ldots \partial_{\mu_n} f(x) \partial_{\nu_1} \ldots \partial_{\nu_n} g(x)$$

Wick-Voros $\star$-product

$$(f \star_V g)(x) = f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial_\mu}\, \theta^{\mu\nu}\, \overrightarrow{\partial_\nu} + \frac{\theta}{2} \overleftarrow{\partial_\mu}\, \overrightarrow{\partial^\mu}\right) g(x)$$

$$T(f \star_M g) = T(f) \star_V T(g), \qquad T = \exp\left(\frac{\theta}{4}\nabla^2\right) \qquad \text{(Berezin, 1971)}$$

Schwartz space of smooth functions of fast decrease

$$S(\mathbb{R}^d) = \left\{ f\colon \sup_x (1+|x|)^N |\partial^\kappa f(x)| < \infty,\ N \in \mathbb{Z}_+\ \kappa \in \mathbb{Z}_+^d \right\}, \qquad \partial^\kappa = \partial_1^{\kappa_1} \cdots \partial_d^{\kappa_d}$$

- The power series defining $\star_M$ and $\star_V$ generally diverge for functions in S
- The Moyal product can be continuously extended from a suitable subspace to S
- For the Wick-Voros product such an extension is impossible

Twisted (Moyal) tensor product

$$(f \otimes_\theta g)(x,y) = f(x) \exp\left(\frac{i}{2}\overleftarrow{\partial_\mu}\,\theta^{\mu\nu}\,\overrightarrow{\partial_\nu}\right) g(y)$$

$$(f \star_\theta g)(x) = (f \otimes_\theta g)(x,x)$$

$$(f \otimes_\theta g) \otimes_\theta h = f \otimes_\theta (g \otimes_\theta h)$$

$$(f \otimes_\theta g)(x_1,\dots,x_m;y_1,\dots,y_n) = \prod_{a=1}^{m}\prod_{b=1}^{n} e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x_a^\mu}\frac{\partial}{\partial y_b^\nu}} f(x_1,\dots,x_m) g(y_1,\dots,y_n)$$

$$(f \,\widehat{\otimes_\theta}\, g)(p,q) = \underbrace{\exp\left(\frac{i}{2}\theta^{\mu\nu}\,p_\mu q_\nu\right)}_{\uparrow}(\hat{f} \otimes \hat{g})(p,q), \qquad f \in S(\mathbb{R}^d),\ g \in S(\mathbb{R}^d)$$

multiplier of $S(\mathbb{R}^{2d})$

Noncommutative deformations of quantum field theories

Let ϕ be a scalar field on commutative Minkowski space and

$$\langle \Psi_0, \phi(f_1) \cdots \phi(f_n) \Psi_0 \rangle = (W^{(n)}, f_1 \otimes \cdots \otimes f_n), \qquad W^{(n)} \in S'(\mathbb{R}^{4n})$$

$$(W_\theta^{(n)}, f_1 \otimes \cdots \otimes f_n) \stackrel{\text{def}}{=} (W^{(n)}, f_1 \otimes_\theta \cdots \otimes_\theta f_n)$$

$$\Phi_n(f) = \int dx_1 \ldots dx_n \, \phi(x_1) \cdots \phi(x_n) f(x_1, \ldots, x_n) \Psi_0, \qquad f \in S(\mathbb{R}^{4n})$$

We set

$$\phi_\theta(g)\Psi_0 = \phi(g)\Psi_0, \qquad \phi_\theta(g)\Phi_n(f) = \Phi_{n+1}(g \otimes_\theta f), \quad n \geq 1$$

Then

$$\langle \Psi_0, \phi(x_1) \cdots \phi(x_n) \Psi_0 \rangle = W_\theta^{(n)}(x_1, \cdots, x_n)$$

and the vacuum state Ψ_0 is cyclic for every field ϕ_θ

Th. 1. **[Phys. Rev. D 77 (2008) 125013]** **Let ϕ be a Hermitian scalar field satisfying the usual Wightman axioms. Then every deformed field ϕ_θ is well-defined as an operator-valued tempered distribution with the same domain in the Hilbert space. Moreover,**

$$\phi_\theta(g)^* = \phi_\theta(\bar{g}) \qquad \textbf{for all} \quad g \in S(\mathbb{R}^4)$$

and

$$\sum_{m,n=1}^{N} (W_\star^{(m+n)}, f_m^\dagger \otimes f_n) \geq 0, \qquad \textbf{for all} \quad f_m \in S(\mathbb{R}^{4m}), f_n \in S(\mathbb{R}^{4n})$$

(with $f^\dagger(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \overline{f(x_n, \ldots, x_1)}$ **)**.

Deformation of a free field

$$a_\theta(p) = e^{(i/2)p\,\theta P} a(p), \qquad a^*_\theta(p) = e^{-(i/2)p\,\theta P} a^*(p),$$

where P is the energy-momentum operator

$$a_\theta(p) a_\theta(p') = e^{-ip\theta p'} a_\theta(p') a_\theta(p)$$

$$a^*_\theta(p) a^*_\theta(p') = e^{-ip\theta p'} a^*_\theta(p') a^*_\theta(p)$$

$$a_\theta(p) a^*_\theta(p') = e^{ip\theta p'} a^*_\theta(p') a_\theta(p) + 2\omega_{\mathbf{p}} \delta(\mathbf{p} - \mathbf{p}')$$

Transformation law of the deformed fields under the Poincaré group

$$U(y,\Lambda)\phi_\theta(x)U^{-1}(y,\Lambda) = \phi_{\Lambda\theta\Lambda^T}(\Lambda x + y), \qquad \Lambda \in L^\uparrow_+$$

Violation of microcausality

If $\Phi = \varphi^{(-)}(h_1)\varphi^{(-)}(h_2)\,\Psi_0$, $h_{1,2} \in S(\mathbb{R}^4)$, then

$$M_\Phi(x,y) = \langle \Psi_0, [\varphi_\theta(x), \varphi_\theta(y)]\,\Phi\rangle \neq 0 \text{ for } (x-y)^2 < 0.$$

Moreover, supp $\widehat{M}_\Phi \subset \bar{V}_+ \times \bar{V}_+ \quad \Longrightarrow \quad$ supp $M_\phi = \mathbb{R}^8$

Localization in wedges

(Grosse, Lechner, JHEP 0809:131, 2008)

$$\theta = \begin{pmatrix} 0 & \theta_e & 0 & 0 \\ -\theta_e & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_m \\ 0 & 0 & -\theta_m & 0 \end{pmatrix} \longrightarrow W_1 = \{x \in \mathbb{R}^4 : x^1 > |x^0|\}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\Lambda\theta\Lambda^T \longleftrightarrow \Lambda W_1$$

If the sets $x + W_\theta$ and $y + W_{\theta'}$ are spacelike separated, then

$$[\phi_\theta(x), \phi_{\theta'}(y)] = 0$$

A causal wedge in place of the light cone

$$\mathcal{O}(x) \stackrel{\text{def}}{=} :\varphi\star\varphi:(x) = \lim_{x_1,x_2\to x} :\varphi(x_1)\varphi(x_2):$$

$$+\sum_{n=1}^{\infty}\left(\frac{i}{2}\right)^n \frac{1}{n!}\,\theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n}\lim_{x_1,x_2\to x} :\partial_{\mu_1}\ldots\partial_{\mu_n}\varphi(x_1)\,\partial_{\nu_1}\ldots\partial_{\nu_n}\varphi(x_2):$$

In the case of space-space noncommutativity ($\theta^{23} = -\theta^{32} \neq 0$ and $\theta^{\mu\nu} = 0$ for $\mu, \nu \neq 2, 3$) the commutator $[\mathcal{O}(x), \mathcal{O}(y)]$ vanishes in the wedge $|x^0 - y^0| < |x^1 - y^1|$, but

$\langle 0|\,[\mathcal{O}(x), \partial_0\mathcal{O}(y)]_-|p_1, p_2\rangle|_{x^0=y^1} \neq 0$ **(Greenberg, Phys. Rev. D, 2006)**

Th. 2. Let $O(x)$ be defined via the Moyal $\star$-product with $\theta^{23} = -\theta^{32} \neq 0$ and the other elements of the θ-matrix equal to zero. Then $[\mathcal{O}(x), \mathcal{O}(y)] \neq 0$ everywhere outside the wedge $|x^0 - y^0| < |x^1 - y^1|$ and the star commutator $[\mathcal{O}(x), \mathcal{O}(y)]_\star \stackrel{\text{def}}{=} \mathcal{O}(x)\star\mathcal{O}(y) - \mathcal{O}(y)\star\mathcal{O}(x)$ does not vanish for all x, y.

The case of Wick-Voros product

Let $\mathcal{O}(x) =: \varphi \star_V \varphi : (x)$, $\theta^{23} = -\theta^{32} = \theta \neq 0$, and $\Phi = \varphi^{(-)}(h_1)\varphi^{(-)}(h_2)\,\Psi_0$. Then

$$\langle \Psi_0, [\mathcal{O}(x), \mathcal{O}(y)]\,\Phi\rangle = 8\int \frac{dk dp_1 dp_2}{(2\pi)^{3(d-1)}}\,\epsilon(k_0)\delta(k^2 - m^2)\,e^{\frac{\theta}{2}\mathbf{k}(\mathbf{p_2}-\mathbf{p_1})}$$

$$\times e^{-ik\cdot(x-y)-ip_1\cdot x-ip_2\cdot y}\prod_{i=1}^{2}\vartheta(p_{i0})\delta(p_i^2 - m^2)\cos\left(\frac{1}{2}k\theta p_i\right)\widehat{h}(p_i),$$

where $\mathbf{k} = (k_2, k_3)$, $k\theta p = k_\mu \theta^{\mu\nu} p_\nu$.

Because of the factor $e^{\frac{\theta}{2}\mathbf{k}(\mathbf{p_2}-\mathbf{p_1})}$ this expression is not a tempered distribution

can be defined only on analytic test functions

Conditions for convergence of the star products

$$(1+|x|)^N|\partial^\kappa f(x)| < C_N B^{|\kappa|}(\kappa!)^{1/2},$$

$$\mathrm{B} < \frac{1}{\sqrt{|\theta|}}, \qquad |\theta| = \sum |\theta^{\mu\nu}|$$

Def. 1. A smooth function f on $\mathbb{R}^d$ belongs to $\mathcal{S}^{1/2}(\mathbb{R}^d)$ if for each $B > 0$ and for any integer N, there exists a constant $C_{B,N}$ such that

$$(1+|x|)^N|\partial^\kappa f(x)| < C_{B,N} B^{|\kappa|}(\kappa!)^{1/2}$$

We endow $\mathcal{S}^{1/2}$ with the topology determined by the set of norms

$$\|f\|_{B,N} = \sup_{x,\kappa}(1+|x|)^N \frac{|\partial^\kappa f(x)|}{B^{|\kappa|}(\kappa!)^{1/2}}$$

Under this topology $\mathcal{S}^{1/2}$ is a nuclear Fréchet space

Test function spaces for NC QFT

Th. 3. [Theor. Math. Phys. 153 (2007); J. Phys. A 40 (2007)]
The space $\mathcal{S}^{1/2}(\mathbb{R}^d)$ is a topological algebra under the Moyal $\star$-product as well as under the Wick-Voros $\star$-product. If $f, g \in \mathcal{S}^{1/2}(\mathbb{R}^d)$, then the series representing these products converge absolutely in this space. Moreover these products depend continuously on the noncommutativity parameter θ.

$\mathcal{S}^{1/2}$ is largest of the subspaces of the Schwartz space that have such properties

We also use the space $S^{1/2,B}_{1/2,A}(\mathbb{R}^d)$ of all smooth functions on $\mathbb{R}^d$ with the property that

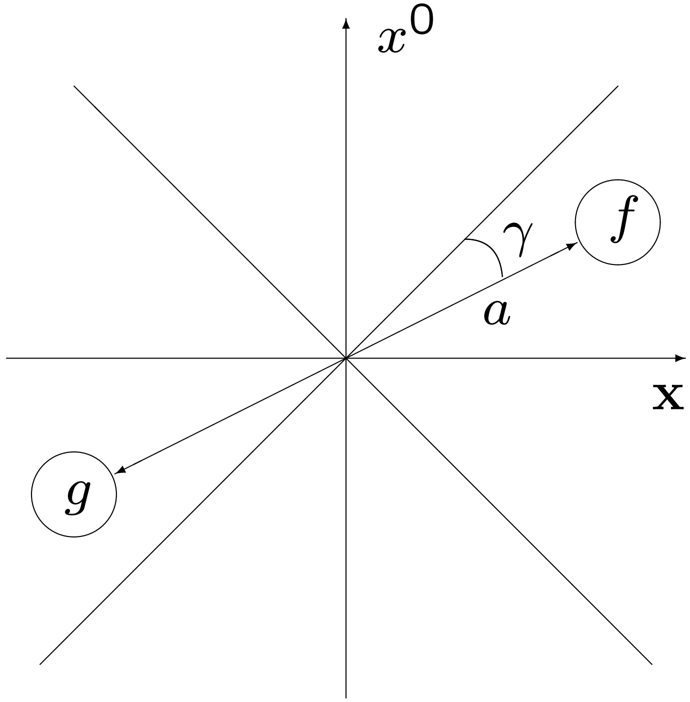
$$\|f\|_{A,B} = \sup_{\kappa,x} e^{|x/A|^2} \frac{|\partial^\kappa f(x)|}{B^{|\kappa|}\kappa^{\kappa/2}} < \infty$$

This space is nontrivial if $AB > 2$

Causal commutator of averaged observables

$$\mathcal{O}(f_a) = \int dx\, \mathcal{O}(x) f(x-a), \qquad f \in S^{1/2,B}_{1/2,A}$$

$$\langle 0 | [\mathcal{O}(f_a), \mathcal{O}(g_{-a})] | \Phi \rangle, \qquad |\Phi\rangle = \varphi^{-}(h_1)\varphi^{-}(h_2)|\Psi_0\rangle$$



$$\gamma = \inf_{\xi^2 \geq 0} \left| \xi - \frac{a}{|a|} \right|$$

Th. 4. **Let ϕ be a free scalar field and let $\mathcal{O}(x) = :\phi \star \phi:(x)$, where the $\star$-product is determined by an arbitrary real antisymmetric matrix $\theta^{\mu\nu}$. Suppose that test functions f and g belong to $S^{1/2,B}_{1/2,A}$, where $A > 0$ and $0 < B < 1/\sqrt{|\theta|}$. Then**

$$|\langle 0|[\mathcal{O}(f_a), \mathcal{O}(g_{-a})]|\Phi\rangle| \leq C_{\Phi,A'} \|f\|_{A,B} \|g\|_{A,B}\, e^{-2|\gamma\, a/A'|^2}$$

for each $A' > A$.

Because of the conditions $B < 1/\sqrt{|\theta|}$ and $AB > 2$, the best result is at $A \sim 2\sqrt{|\theta|}$ and demonstrates a decrease

$$\sim \exp\left(-\frac{|\gamma a|^2}{2|\theta|}\right)$$

The same result holds in the case of Wick-Voros product

Def. 2. Let V be a cone in $\subset \mathbb{R}^d$. A smooth function on $\mathbb{R}^d$ belongs to the space $S^{1/2,B}(V)$, if it satisfies the condition

$$\sup_{x \in V} (1+|x|)^N |\partial^\kappa f(x)| < C_N B^{|\kappa|} (\kappa!)^{1/2},$$

$$\mathbb{V} = \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 \colon (x-y)^2 \geq 0\}.$$

θ-locality condition: **For any fields φ, ψ and for any states Φ, Ψ in a common invariant domain $D \subset H$, either**

$$\langle \Phi\,, [\phi(x), \psi(x')]_- \Psi \rangle$$

or

$$\langle \Phi\,, [\phi(x), \psi(x')]_+ \Psi \rangle$$

can be continuously extended to the space $S^{1/2,B}(\mathbb{V})$, where $B_{\varphi,\Psi,\Phi,\Psi} \sim 1/\sqrt{|\theta|}$.

Conclusions

• The noncommutative deformation of QFT by twisting tensor products leads to the lack of microcausality, though preserves (in the Moyal case) certain relative localization properties

• The space $\mathcal{S}^{1/2}$ is universal for a nonperturbative (in particular for a Wightman-type axiomatic) formulation of NC QFT

• This space, being a maximal topological star product algebra with absolute convergence, is completely adequate to the concept of strict deformation quantization

• The θ-locality condition, which means heuristically that the commutators of observables behave at large spacelike separation like

$$\exp(-|x-y|^2/\theta)$$

can possibly be used for formulating causality in NC QFT