Non-perturbative treatment of homogeneous non-Gaussian integrals

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There

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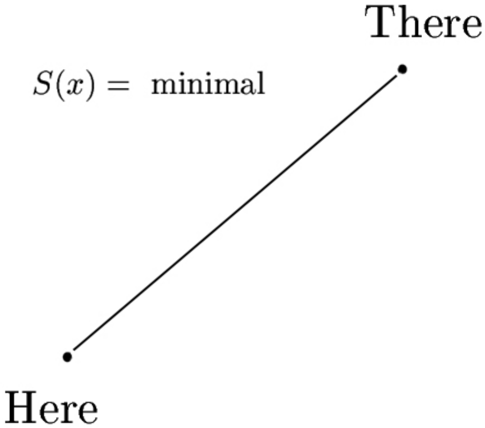
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$S(x) =$ minimal
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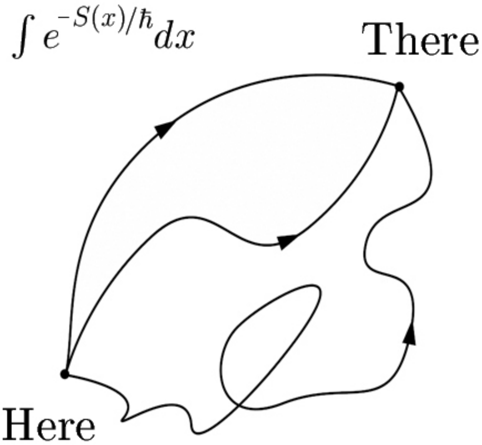
$\int e^{-S(x)/\hbar} dx$
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Probabilistic character of the theory

To compute any average, we calculate an integral

$$\left\langle f(x) \right\rangle = \int f(x) e^{-S(x)} dx$$

Do we know, how to calculate such integrals?

The Gaussian integration formula

We do, if $S(x)$ is quadratic:

$$\int e^{-S_{ij}x_i x_j} d^n x = \frac{1}{\sqrt{\det S}}$$

Non-Gaussian integration formula?

If $S(x)$ is cubic or higher, much less is known:

$$\int e^{-S_{ijk}x_i x_j x_k} d^n x \;\; = ?$$

More generally:

Homogeneous form of degree r in n variables:

$$S(x_1, \ldots, x_n) = S_{i_1,\ldots,i_r} x_{i_1} \ldots x_{i_r}$$

Homogeneous non-Gaussian integral of type $n|r$:

$$J_{n|r}(S) = \int e^{-S(x_1,\ldots,x_n)} d^n x \ = ?$$

Scaling symmetry

$$J_{n|r}(\lambda S) = \int e^{-\lambda S(x_1,\ldots,x_n)} d^n x \ = \ \lambda^{-n/r} J_{n|r}(S)$$

$SL(n)$ symmetry

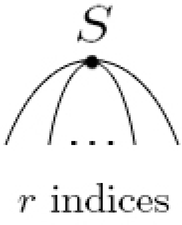
$$J_{n|r}(S) \;=\; SL(n) \text{ invariant function of } S_{i_1,\ldots,i_r}$$

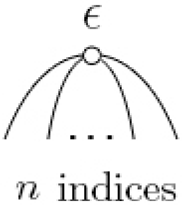
$$\text{Say, } J_{n|2}(S) \;= \frac{1}{\sqrt{\det S}} - \text{ invariant}$$

$SL(n)$ invariants

All $SL(n)$ invariants of a form S of type $n|r$ can be

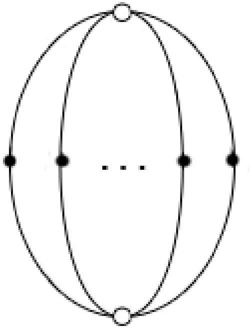
represented as diagrams, made of S-vertices and ϵ-vertices:





Determinant of a matrix

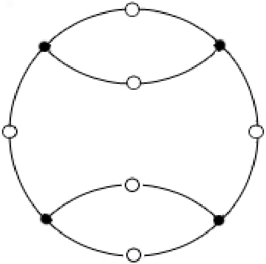
Say, determinant of $n \times n$ matrix looks like



The # of independent invariants I_k of a form of type $n|r$

$r \backslash n$	2	3	4	5	6	7
2	1	1	1	1	1	1
3	1	2	5	11	21	36
4	2	7	20	46	91	162
5	3	13	41	102	217	414
6	4	20	69	186	427	876

There is a single independent invariant, called "discriminant":



$$D = 27a^2d^2 - b^2c^2 - 18abcd + 4ac^3 + 4b^3d$$

Case 2|3

Therefore, the integral must be a function of D:

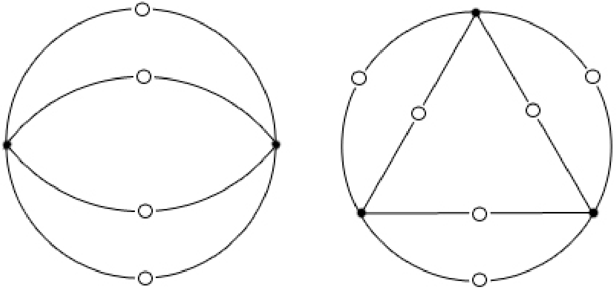
$$J_{2|3}(S) = F(D)$$

Scaling symmetry implies that

$$J_{2|3}(S) = \frac{1}{\sqrt[6]{D}}$$

Case 2|4: form $S(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$

There are 2 independent invariants, called apolara and Hankel invariant:



Case 2|4

$$I_2 = c^2 - 3bd + 12ae$$

$$I_3 = 2c^3 - 9bcd + 27b^2e + 27ad^2 - 72ace$$

Case 2|4

Therefore, the integral must be a function of I_2, I_3:

$$J_{2|4}(S) = F(I_2, I_3)$$

Scaling symmetry is no longer powerful:

$$J_{2|4}(S) = \frac{1}{\sqrt[4]{I_2}} G\left(\frac{I_3^2}{I_2^3}\right)$$

We need something else to determine the function $G(z)$.

Case 2|4: a differential equation

Differential equations can be helpful. Integral

$$J_{2|4} = \int e^{-(ax^4+bx^3y+cx^2y^2+dxy^3+ey^4)}dxdy$$

satisfies a differential equation

$$\left(\frac{\partial}{\partial a}\frac{\partial}{\partial c} - \frac{\partial}{\partial b}\frac{\partial}{\partial b}\right) J_{2|4} = 0$$

Case 2|4: complete system of equations

$$\left(\frac{\partial}{\partial a}\frac{\partial}{\partial c} - \frac{\partial}{\partial b}\frac{\partial}{\partial b}\right) J_{2|4} = 0$$

$$\left(\frac{\partial}{\partial a}\frac{\partial}{\partial d} - \frac{\partial}{\partial b}\frac{\partial}{\partial c}\right) J_{2|4} = 0$$

$$\left(\frac{\partial}{\partial a}\frac{\partial}{\partial e} - \frac{\partial}{\partial b}\frac{\partial}{\partial d}\right) J_{2|4} = 0$$

$$\left(\frac{\partial}{\partial a}\frac{\partial}{\partial e} - \frac{\partial}{\partial c}\frac{\partial}{\partial c}\right) J_{2|4} = 0$$

Case 2|4: hypergeometric equation

If we substitute our ansatz

$$J_{2|4}(S) = \frac{1}{\sqrt[4]{I_2}} G\left(\frac{I_3^2}{I_2^3}\right)$$

these equations are translated into single equation on $G(z)$:

$$(144z^2 - 24z)\frac{\partial^2 G(z)}{\partial z^2} + (216z - 12)\frac{\partial G(z)}{\partial z} + 5G(z) = 0$$

Case 2|4: hypergeometric function

$$(144z^2 - 24z)\frac{\partial^2 G(z)}{\partial z^2} + (216z - 12)\,\frac{\partial G(z)}{\partial z} + 5G(z) = 0$$

This is a classical hypergeometric equation. Accordingly,

$$J_{2|4}(S) = \frac{1}{\sqrt[4]{I_2}}\;{}_2F_1\left(\left[\frac{1}{12},\frac{5}{12}\right],\left[\frac{1}{2}\right],\frac{6I_3^2}{I_2^3}\right)$$

Case 2|4: series solution

Using the Pochhammer symbol $(a)_k = a(a+1)\dots(a+k-1)$, we can write

$$J_{2|4}(S) = I_2^{-1/4} \cdot \sum_{i=0}^{\infty} \frac{(1/12)_i (5/12)_i}{(1/2)_i} \; \frac{u^i}{i!}$$

where $u = \dfrac{6I_3^2}{I_2^3}$ is the dimensionless ratio

Case 2|4: there is a singularity!

$$J_{2|4}(S) = \frac{1}{\sqrt[4]{I_2}}\ {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], \left[\frac{1}{2}\right], \frac{6I_3^2}{I_2^3}\right)$$

The singularity resides at z = 1, i.e, $I_2^3 - 6I_3^2 = 0$

Amazingly, $D = I_2^3 - 6I_3^2$ is exactly the discriminant of $S(x, y)$

What is discriminant?

For any homogeneous form $S(x_1, \ldots, x_n) = S_{i_1,\ldots,i_r} x_{i_1} \ldots x_{i_r}$

the system of derivatives is solvable

$$\begin{cases} \dfrac{\partial S}{\partial x_1} = 0 \\ \quad \ldots \\ \dfrac{\partial S}{\partial x_n} = 0 \end{cases}$$

if and only if coefficients S satisfy some condition $D(S) = 0$.

Hypothesis:

Singularities of non-Gaussian integrals

$$J_{n|r}(S) = \int e^{-S(x_1,\ldots,x_n)} d^n x$$

are controlled by discriminant of S

For this reason, J can be naturally called integral discriminants

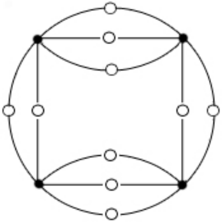
Case 2|5: a form of degree 5 in 2 variables

The form looks like

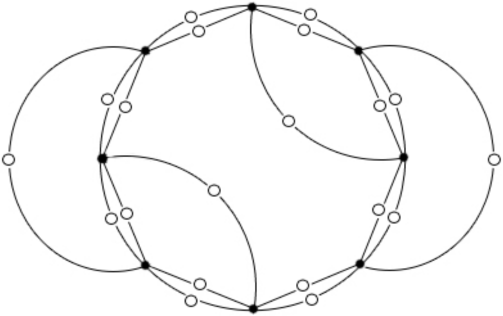
$$S(x,y) = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5$$

There are 3 independent invariants in this case

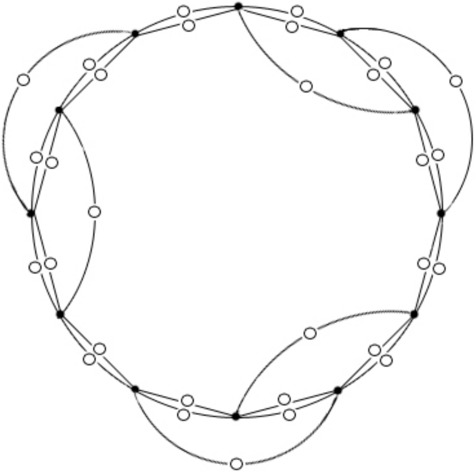
Case 2|5: invariant of degree 4



Case 2|5: invariant of degree 8



Case 2|5: invariant of degree 12



The answer

Applying the same procedure with differential equations, one ends with

$$J_{2|5}(S) = I_4^{-1/10} \cdot \sum_{i,j=0}^{\infty} \frac{(3/10)_{i+j}(1/10)_{2i+3j}(1/10)_j}{(2/5)_{i+2j}(3/5)_{i+2j}} \; \frac{u^i}{i!}\frac{v^j}{j!}$$

where $u = \dfrac{16 I_8}{I_4^2}$ and $v = \dfrac{128 I_{12}}{3 I_4^3}$ are the dimensionless ratios

Case 3|3: a form of degree 3 in 3 variables

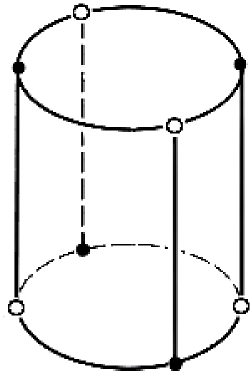
The form looks like

$$S(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2z +$$

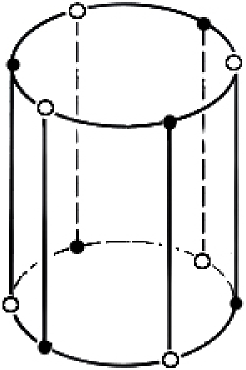
$$+fxyz + gy^2z + hxz^2 + pyz^2 + qz^3$$

There are 2 independent invariants in this case

Case 3|2: invariant of degree 4



Case 3|2: invariant of degree 6



The answer

Applying the same procedure with differential equations, one ends with

$$J_{3|3}(S) = I_4^{-1/4} \cdot \sum_{i=0}^{\infty} \frac{(1/12)_i (5/12)_i}{(1/2)_i} \; \frac{u^i}{i!}$$

where $u = -\dfrac{3I_6^2}{32I_4^3}$ is the dimensionless ratio

Conclusion

n	r	Integral discriminant $J_{n\|r}$
2	3	$I_4^{-1/6}$
2	4	$I_2^{-1/4} \cdot \sum\limits_{i=0}^{\infty} \frac{1}{i!} \cdot \frac{(1/12)_i(5/12)_i}{(1/2)_i} \cdot \left(\frac{6I_3^2}{I_2^3}\right)^i$
2	5	$I_4^{-1/10} \cdot \sum\limits_{i,j=0}^{\infty} \frac{1}{i!j!} \cdot \frac{(3/10)_{i+j}(1/10)_{2i+3j}(1/10)_j}{(2/5)_{i+2j}(3/5)_{i+2j}} \cdot \left(\frac{16I_8}{I_4^2}\right)^i \left(\frac{128I_{12}}{3I_4^3}\right)^j$
3	3	$I_4^{-1/4} \cdot \sum\limits_{i=0}^{\infty} \frac{1}{i!} \cdot \frac{(1/12)_i(5/12)_i}{(1/2)_i} \cdot \left(-\frac{3I_6^2}{32I_4^3}\right)^i$
...	...	...

Thank you very much for your attention!