Pure Gauge Configurations and Solutions to Fermionic Superstring Field Theories Equations of Motion

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based on
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Cubic Superstring Field Theory
Aref'eva, Medvedev, Zubarev
Preitschopf, Thorn, Yost

- Action

$$S = \frac{1}{2}\langle Y_{-2}\Phi, Q\Phi\rangle + \frac{1}{3}\langle Y_{-2}\Phi, \Phi \star \Phi\rangle$$

Aref'eva, Medvedev, Zubarev (1990)

Preitschopf, Thorn, Yost (1990)

  - $\Phi \equiv \Phi[X^{\mu}, c, b, \psi^{\mu}, \beta, \gamma]$ – string functional,
  - $\langle \cdot, \cdot \rangle$ – bilinear inner product,
  - Y_{-2} – double-step picture changing operator,
  - Q – BRST charge,

    $Q^2 = 0, \quad Q(\Phi_1 \star \Phi_2) = Q\Phi_1 \star \Phi_2 + (-)^{|\Phi_1|}\Phi_1 \star Q\Phi_2$

The action is invariant under the gauge transformation

$$\Phi \to U^{-1}(\Phi + Q)U$$

- Equation of motion

$$Y_{-2}(Q\Phi + \Phi \star \Phi) = 0 \to Q\Phi + \Phi \star \Phi = 0$$

Cubic Superstring Field Theory

$$\Phi = \lim_{\lambda \to 1} \Phi_\lambda$$

- We shall solve the equation of motion perturbatively in a parameter λ

$$\Phi_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1} \phi_n$$

  - At order λ we find $Q\phi_0 = 0$. Let us suppose that

$$\phi_0 = Q\phi$$

  - At the second order in λ

$$Q\phi_1 + \phi_0 \star \phi_0 = Q\phi_1 + Q\phi \star \phi = Q(\phi_1 - Q\phi \star \phi) = 0.$$

    Solution is

$$\phi_1 = Q\phi \star \phi.$$

  - At the n-order in λ

$$\phi_{n-1} = Q\phi \star \phi^{n-1}.$$

Solution as a pure gauge configuration

- Solution

$$\Phi_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1} Q\phi \star \phi^n = \lambda Q\phi \frac{1}{1-\lambda\phi}$$

If we denote $U_\lambda \equiv \frac{1}{1-\lambda\phi}$ we can write down Φ_λ as

$$\Phi_\lambda = -(QU_\lambda^{-1})U_\lambda = U_\lambda^{-1} Q U_\lambda .$$

- Initial date

$$\phi = B_1^L c_1|0\rangle, \qquad B_1^L = \int_{C_L} \frac{dz}{2\pi i}(z^2+1)b(z),$$

$$\phi^n = |n\rangle \star \phi, \qquad |n\rangle = \underbrace{|0\rangle \star \ldots \star |0\rangle}_{n-1}$$

- Therefore Φ_λ is equal

$$\Phi_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1} \partial_n \varphi_n + \lambda \Gamma, \qquad \textit{Erler } 2007$$

where

$$\varphi_n = c_1|0\rangle \star |n\rangle \star B_1^L c_1 |0\rangle, \quad \text{and} \quad \Gamma = B_1^L \gamma_{\frac{1}{2}} |0\rangle.$$

- Regulated solution

$$\Phi = \lim_{N\to\infty} \left(\sum_{n=0}^{N} \partial_n \varphi_n - \varphi_N - \frac{1}{2} \partial_N \varphi_N \right) + \Gamma \qquad \textit{Erler } 2007$$

  - $\langle\langle \Phi, Q\Phi + \Phi \star \Phi \rangle\rangle = 0$

- Partial sum
$$\Phi_\lambda^N = \sum_{n=0}^{N} \lambda^{n+1}\partial_n\varphi_n + \lambda\Gamma$$
we check a validity of the equation of motion in a weak since
$$\langle\langle\varphi_m, Q\Phi_\lambda^N + \Phi_\lambda^N \star \Phi_\lambda^N\rangle\rangle$$
- Correlators (Erler 2007)
$$\langle\langle\varphi_m, Q\varphi_n\rangle\rangle = -\frac{m+n+2}{\pi^2}, \quad \langle\langle\varphi_m, Q\Gamma\rangle\rangle = \frac{1}{\pi^2},$$
$$\langle\langle\Gamma, \varphi_m \star \varphi_n\rangle\rangle = \frac{m+n+3}{2\pi^2}$$
weak since
- $\langle\langle\varphi_m, Q\Phi_\lambda^N + \Phi_\lambda^N \star \Phi_\lambda^N\rangle\rangle = \frac{\lambda^{N+1}}{\pi^2}$

- Regularization

  $\Phi^N_R(a,b) = \Phi^N + a\varphi_N + b\varphi'_N, \quad \text{where} \quad \Phi^N \equiv \sum_{n=0}^{N-1} \varphi'_n + \Gamma$

  - $\langle\langle \varphi_m, Q\Phi^N_R(a,b) + \Phi^N_R(a,b) \star \Phi^N_R(a,b) \rangle\rangle = \frac{1}{\pi^2} + a\frac{1}{\pi^2}$

$$a = -1$$

  - $\langle\langle \Gamma, Q\Phi^N_R(-1,b) + \Phi^N(-1,b)_R \star \Phi^N_R(-1,b) \rangle\rangle = \frac{1}{2\pi^2} - b\frac{1}{\pi^2}$

$$b = \frac{1}{2}$$

- Solution

$$\Phi^N_R(-1,1/2) = \sum_{n=0}^{N-1} \varphi'_n + \Gamma - \varphi_N + \frac{1}{2}\varphi'_N$$

Action

- $$\langle\langle \Phi_R^N, Q\Phi_R^N + \Phi_R^N \star \Phi_R^N \rangle\rangle = -\frac{a}{\pi^2}\left[(1+a)N + \frac{5}{2}a + b + 3\right]$$

- Action

$$S = \frac{1}{2}\langle\langle \Phi_R^N, Q\Phi_R^N \rangle\rangle + \frac{1}{3}\langle\langle \Phi_R^N, \Phi_R^N \star \Phi_R^N \rangle\rangle = -\frac{a(2+a)}{2\pi^2}$$

$$S = \frac{1}{2\pi^2}$$

- Action for NS sector of the cubic fermionic string field theory (ABKM arXiv:hep-th/0011117)

$$S[\widehat{\Phi}] = Tr\left[\frac{1}{2}\langle\widehat{Y}_{-2}\widehat{\Phi}, \widehat{Q}\widehat{\Phi}\rangle + \frac{1}{3}\langle\widehat{Y}_{-2}\widehat{\Phi}, \widehat{\Phi}\star\widehat{\Phi}\rangle\right]$$

  - $\widehat{\Phi} = \Phi_{+}\otimes\sigma_3 + \Phi_{-}\otimes i\sigma_2$ – string field,
  - $\widehat{Q} = Q\otimes\sigma_3, \quad \widehat{Y}_{-2} = Y_{-2}\otimes\sigma_3,$
    - $\widehat{Q}^2 = 0,$
    - $\widehat{Q}(\widehat{\Phi}\star\widehat{\Psi}) = (\widehat{Q}\widehat{\Phi})\star\widehat{\Psi} + (-)^{|\widehat{\Phi}|}\widehat{\Phi}\star(\widehat{Q}\widehat{\Psi})$
- Equation of motion

$$\widehat{Q}\widehat{\Phi} + \widehat{\Phi}\star\widehat{\Phi} = 0$$

- Action for the NS sector of the non-polynomial fermionic string field theory (Berkovits arXiv:hep-th/0001084)

$$\begin{aligned} S[\hat{\Psi}] &= \frac{1}{4}Tr\int\Big[(e^{-\hat{\Psi}}\hat{Q}e^{\hat{\Psi}})(e^{-\hat{\Psi}}\hat{\eta}_0e^{\hat{\Psi}}) \\ &\quad - \int_0^1 dt\,(e^{-t\hat{\Psi}}\partial_t e^{t\hat{\Psi}})\{(e^{-t\hat{\Psi}}\hat{Q}e^{t\hat{\Psi}}), (e^{-t\hat{\Psi}}\hat{\eta}_0e^{t\hat{\Psi}})\}, \end{aligned}$$

$$\hat{\Psi} = \Psi_+ \otimes I + \Psi_- \otimes \sigma_1 - \text{string field}$$

$$\hat{\eta}_0 = \eta_0 \otimes \sigma_3$$

- Equation of motion

$$\widehat{\eta}_0(\widehat{G}^{-1}\widehat{Q}\widehat{G}) = 0, \qquad \widehat{G} = e^{\widehat{\Psi}}$$

$$\widehat{G} = G_+ \otimes I + G_- \otimes \sigma_1$$

where

$$\begin{aligned} G_+ &= I + \Psi_+ + \frac{1}{2}\Psi_+ \star \Psi_+ + \frac{1}{2}\Psi_- \star \Psi_- + \ldots \\ G_- &= \Psi_- + \frac{1}{2}\Psi_+ \star \Psi_- + \frac{1}{2}\Psi_- \star \Psi_+ + \ldots \end{aligned}$$

Equivalence

- Let $\mathfrak{A}$ be a set of matrix solutions to equation of motion ABKM theory and $\mathfrak{B}$ be a set of solutions to Berkovits$\pm$ theory
  - map g of $\mathfrak{B}$ to $\mathfrak{A}$

$$g : \widehat{G} \to \widehat{\Phi} \equiv g(\widehat{G}) = \widehat{G}^{-1}\widehat{Q}\widehat{G}$$

  - map h of $\mathfrak{A}$ to $\mathfrak{B}$

$$h : \widehat{\Phi} \to \widehat{G} \equiv h(\widehat{\Phi}) = e^{\widehat{P}\widehat{\Phi}}$$

$\widehat{P} \equiv P \otimes \sigma_3$, $(P\Phi_1) \star (P\Phi_2) = 0$, $\{Q, P\} = 1$ (Fuchs, Kroyter arXiv:0805.4386)

$$\widehat{G} = e^{\widehat{P}\widehat{\Phi}} = 1 + \widehat{P}\widehat{\Phi},$$

In the components

$$G_+ = 1 + P\Phi_+ = e^{P\Phi_+}, \quad G_- = P\Phi_-.$$

- Composition $g \circ h$

$$\begin{aligned}\widetilde{\widehat{\Phi}} &= (g \circ h)(\widehat{\Phi}) = g(h(\widehat{\Phi})) = (1 - \widehat{P}\widehat{\Phi})\widehat{Q}(1 + \widehat{P}\widehat{\Phi}) \\ &= (1 - \widehat{P}\widehat{Q} - \widehat{P}\widehat{\Phi})\widehat{\Phi} = \widehat{\Phi} - \widehat{P}(\widehat{Q}\widehat{\Phi} + \widehat{\Phi}^2) = \widehat{\Phi},\end{aligned}$$

  - We have proved that $g \circ h = Id$ and $g(\mathfrak{B}) = \mathfrak{A}$
- Composition $h \circ g$

$$\begin{aligned}\widetilde{\widehat{G}} &= (h \circ g)(\widehat{G}) = h(g(\widehat{G})) = e^{\widehat{P}\widehat{G}^{-1}\widehat{Q}\widehat{G}} = 1 + \widehat{P}\widehat{G}^{-1}\widehat{Q}\widehat{G} \\ &= 1 - (1 - \widehat{Q}\widehat{P})\widehat{G}^{-1} \cdot \widehat{G} = \widehat{Q}\widehat{P}\widehat{G}^{-1} \cdot \widehat{G}.\end{aligned}$$

- Parametrization

$$\widehat{G} = \frac{1}{1 - \widehat{\phi}}.$$

$$\widehat{\Phi} = \widehat{G}^{-1}\widehat{Q}\widehat{G} = -\widehat{Q}\widehat{G}^{-1}\widehat{G} = \widehat{Q}\widehat{\phi}\frac{1}{1 - \widehat{\phi}}.$$

$$\widetilde{\widehat{G}} = \widehat{Q}\widehat{P}\widehat{G}^{-1}\cdot\widehat{G} = \widehat{Q}\widehat{P}(1-\widehat{\phi})\cdot\frac{1}{1-\widehat{\phi}} = (1-\widehat{Q}(\widehat{P}\widehat{\phi}))\widehat{G} = e^{-\widehat{Q}(\widehat{P}\widehat{\phi})}\widehat{G},$$

$$\begin{aligned} \widetilde{G}_+ &= e^{-Q\Lambda_+}G_+ - Q\Lambda_- G_-, \\ \widetilde{G}_- &= e^{-Q\Lambda_+}G_- - Q\Lambda_- G_+ \end{aligned}$$

- Gauge transformation

$$\widetilde{\widehat{G}} = e^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}},$$

gauge parameter $\widehat{\Lambda}_{\widehat{Q}} = \widehat{P}\widehat{\phi}$, $\widehat{\Lambda}_{\widehat{\eta}} = 0$.

Assertion

$(h \circ g)(\widehat{G})$ belongs to a gauge orbit $\mathfrak{O}_{\widehat{G}} = \{\widetilde{\widehat{G}} : \widetilde{\widehat{G}} = e^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}\}$ of the initial field $\widehat{G}$

- Image of the orbit $\mathfrak{O}_{\widehat{\Phi}} = \{\widetilde{\widehat{\Phi}} : \widetilde{\widehat{\Phi}} = e^{-\widehat{\Lambda}}(\widehat{\Phi} + \widehat{Q})e^{\widehat{\Lambda}}\}$ by the map h: $h(\mathfrak{O}_{\widehat{\Phi}}) = \{\widetilde{\widehat{G}} : \widetilde{\widehat{G}} = h(\widetilde{\widehat{\Phi}})\}$.

$$
\begin{aligned}
\widetilde{\widehat{G}} &= 1 + \widehat{P}\widetilde{\widehat{\Phi}} = 1 + \widehat{P}(e^{-\widehat{\Lambda}}(\widehat{\Phi} + \widehat{Q})e^{\widehat{\Lambda}}) \\
&= e^{-\widehat{Q}\widehat{P}\widehat{\Lambda}}\widehat{G}e^{\widehat{\Lambda}} = e^{-\widehat{Q}\widehat{P}\widehat{\Lambda}}\widehat{G}e^{\widehat{\eta}_0\widehat{\xi}\widehat{\Lambda}},
\end{aligned}
$$

- Image of the orbit $\mathfrak{O}_{\widehat{G}} = \{\widetilde{\widehat{G}} : \widetilde{\widehat{G}} = e^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}\}$ by the map g: $g(\mathfrak{O}_{\widehat{G}}) = \{\widetilde{\widehat{\Phi}} = g(\widetilde{\widehat{G}})\}$

$$
\begin{aligned}
\widetilde{\widehat{\Phi}} &= \widetilde{\widehat{G}}^{-1}\widehat{Q}\widetilde{\widehat{G}} = e^{-\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}\widehat{G}^{-1}e^{\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{Q}(e^{-\widehat{Q}\widehat{\Lambda}_{\widehat{Q}}}\widehat{G}e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}) \\
&= e^{-\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}\widehat{G}^{-1}((\widehat{Q}\widehat{G})e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}} + \widehat{G}\widehat{Q}e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}) = e^{-\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}(\widehat{\Phi} + \widehat{Q})e^{\widehat{\eta}_0\widehat{\Lambda}_{\widehat{\eta}}}
\end{aligned}
$$

Assertion

The image $h : \mathfrak{O}_{\widehat{\Phi}} \to \mathfrak{O}_{\widehat{G}}$ *is suborbit. The image* $g : \mathfrak{O}_{\widehat{G}} \to \mathfrak{O}_{\widehat{\Phi}}$ *is all orbit. All elements* $\mathfrak{O}_{\widehat{G}}$ *with different* $\widehat{\Lambda}_{\widehat{Q}}$ *are mapped in one element* $\mathfrak{O}_{\widehat{\Phi}}$. *Bounded on* $h(\mathfrak{O}_{\widehat{\Phi}})$ *mapping* g *becomes invertible:* $h \circ g|_{h(\mathfrak{O}_{\widehat{\Phi}})} = Id$. *The composition* $h \circ g$ *gives in the orbit* $\mathfrak{O}_{\widehat{G}}$ *a special section.*